Math for Management, Winter 2023

List 9

More functions of two variables

230. Give the partial derivative of

$$z = f(x,y) = xy^3 + x^5\, e^{xy} - 2^x$$

with respect to x, which is a new function with two inputs. We can write any of

$$f'_x(x,y) \qquad f_x(x,y) \qquad z'_x(x,y) \qquad z_x(x,y) \qquad f'_x \qquad f_x \qquad z'_x \qquad z_x$$

for this function.
It may help to think about $(ax + x^5\, e^{b\,x} - 2^x)'$, where a, b, c are constants.

$\boxed{y^3 + x^5 y e^{xy} + 5x^4 e^{xy} - 2^x \ln(2)}$

231. Give the partial derivative of

$$z = f(x,y) = xy^3 + x^5\, e^{xy} - 2^x$$

with respect to y, which is a new function with two inputs. We can write any of

$$f'_y(x,y) \qquad f_y(x,y) \qquad z'_y(x,y) \qquad z_y(x,y) \qquad f'_y \qquad f_y \qquad z'_y \qquad z_y$$

for this function.
It may help to think about $(at^3 + b\, e^{c\,t} - d)'$, where a, b, c, d are constants.

$\boxed{3xy^2 + x^6 e^{xy}}$

232. (a) Calculate the partial derivative of $f(x,y) = y^x$ with respect to x, which is a function. We can write $f'_x(x,y)$ or $f_x(x,y)$ or f'_x or f_x for this. $\boxed{f'_x = y^x \ln(y)}$
   (b) Calculate the partial derivative of $f(x,y) = y^x$ with respect to y, which is a function. We can write $f'_y(x,y)$ or $f_y(x,y)$ or f'_y or f_y for this. $\boxed{f'_y = xy^{x-1}}$
   (c) Calculate the partial derivative of $f(x,y) = y^x$ with respect to x at the point $(5,2)$, which is a number. We can write $f'_x(5,2)$ or $f_x(5,2)$ for this. $\boxed{\ln(9) \approx 2.19722}$
   (d) Calculate the partial derivative of $f(x,y) = y^x$ with respect to y at the point $(5,2)$, which is a number. We can write $f'_y(5,2)$ or $f_y(5,2)$ for this. $\boxed{5 \cdot 2^4 = 80}$

233. For the function $z = x^8 y^2$, calculate
   (a) $z_x = \boxed{8x^7y^2}$
   (b) $z_{xx} = (z_x)_x = \boxed{56x^6y^2}$
   (c) $z_{xy} = (z_x)_y = \boxed{16x^7y}$
   (d) $z_y = \boxed{2x^8y}$
   (e) $z_{yx} = (z_y)_x = \boxed{16x^7y}$
   (f) $z_{yy} = (z_y)_y = \boxed{2x^8}$

(g) $z_{xx}\, z_{yy} - z_{xy}\, z_{yx} \;= (56x^6y^2)(2x^8) - (16x^7y)^2 = \boxed{-144x^{14}y^2}$

The point $(x, y) = (a, b)$ is a **stationary point** of $f(x, y)$ if both $f'_x(a, b) = 0$ and $f'_y(a, b) = 0$, where f'_x and f'_y are partial derivatives of f. A **critical point** is where either $f'_x = f'_y = 0$ or at least one partial d. does not exist.

234. Find the stationary points of

$$f(x, y) = 2x^2(x - \tfrac{3}{2}y - 6) - 3e^{2\ln(y)}.$$

Hint: there are three. $\boxed{(0, 0),\ (-4, -8),\ (2, -2)}$

235. Find all stationary points for each of the following functions.

(a) $f(x, y) = e^{7x} - xy \begin{cases} 7e^{7x} - y = 0 \\ -x = 0 \end{cases}$ when $(x, y) = \boxed{(0, 7)}$ only.

(b) $z = x^3 + 8y^3 - 3xy$ $\boxed{(0, 0) \text{ and } (\tfrac{1}{2}, \tfrac{1}{4})}$

(c) $f = y\ln(x^2) + x$ $\boxed{(1, -\tfrac{1}{2}) \text{ and } (-1, \tfrac{1}{2})}$

The function $D(x, y) = f''_{xx}f''_{yy} - f''_{xy}f''_{yx}$ can be used to classify critical points.
If $D > 0$ and $f''_{xx} > 0$ at a critical point, then that point is a local minimum.
If $D > 0$ and $f''_{xx} < 0$ at a critical point, then that point is a local maximum.
If $D < 0$ at a critical point then it is *not* a local extreme (it is a "saddle").
If $D = 0$ then the point might be a local extreme but might not be.

236. Find and classify all the critical points of $f(x, y) = 2x^2(x - \frac{3}{2}y - 6) - 3e^{2\ln(y)}$.

See Task 234. CPs are $(0, 0)$, $(-4, -8)$, $(2, -2)$.

$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (12x - 6y - 24)(-6) - (-6x)(-6x) = 36(y - 2x - x^2 + 4)$.

x	y	D	f''_{xx}	
0	0	$+$	$-$	$\boxed{\text{local max at } (0, 0)}$
-4	-8	$-$		saddle
2	-2	$-$		saddle

To find extreme values of $f(x, y)$ with a condition/restriction, re-write the task as a single-variable extreme value task (see List 6 for those kinds of tasks).
For extreme values of $f(x, y)$ in a polygonal domain, check the value of the function at all critical points inside the domain and at all vertices (corners) of the domain.

237. Find the smallest and largest values of

$$f(x, y) = 9x^2 - 6x - y^3 - y^2 + 9$$

on the filled square $0 \le x \le 1$, $0 \le y \le 1$ by following these steps:

(a) Find the critical points of $f(x, y)$. Ignore any that do not satisfy $0 \le x \le 1$ and $0 \le y \le 1$.

(b) Find single-variable-critical-points on the boundary of the square by looking at each side separately.

(c) Compute the value of f at all points from steps (a) and (b). The largest f-value is the maximum and the most negative f-value is the minimum.

$f'_x = 18x - 6$ and $f'_y = -3y^2 - 2y$. Solving $\begin{cases} 18x - 6 = 0 \\ -3y^2 - 2y = 0 \end{cases}$ gives $(x, y) = (\frac{1}{3}, -\frac{2}{3})$, which is NOT in the square, and $(\frac{1}{3}, 0)$, which is.

Bottom: $y = 0$. $f = 9x^2 - 6x + 9$ has $f' = 18x - 6 = 0$ when $x = \frac{1}{3}$, which is $(x, y) = (\frac{1}{3}, 0)$ again.

Top: $y = 1$. $f = 9x^2 - 6x + 7$ has $f' = 18x - 6 = 0$ when $x = \frac{1}{3}$, which is $(x, y) = (\frac{1}{3}, 1)$ now.

Left: $x = 0$. $f = 9 - y^2 - y^3$ has $f' = -2y - 3y^2 = 0$ when $y = 0$ or $y = -\frac{2}{3}$. These are $(0, 0)$ and $(0, -\frac{2}{3})$, the second of which is not in the square.

Right: $x = 1$. $f = 12 - y^2 - y^3$ has $f' = -2y - 3y^2 = 0$ when $y = 0$ or $y = -\frac{2}{3}$. These are $(1, 0)$ and $(1, -\frac{2}{3})$, the second of which is not in the square.

We also need to check the corners $(0, 1)$ and $(1, 1)$ that have not already been mentioned.

x	y	f
1/3	0	8
1/3	1	$\boxed{6 \text{ min}}$
0	0	9
1	0	$\boxed{12 \text{ max}}$
0	1	7
1	1	10

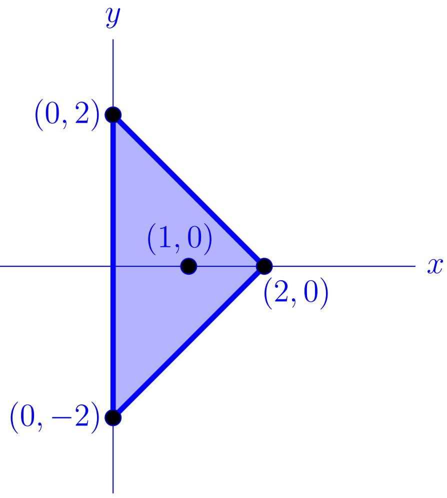
238. Find the extreme values of

$$z = (x - 3)^2 + (y + 1)^2 - 2(y + 5 - 2x)$$

on the solid triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.

$z = x^2 + y^2 - 2x$ after expanding. Call this $f(x, y)$.

$f'_x = 2x - 2$ and $f'_y = 2y$. The only critical point inside the triangle is at $(1, 0)$.



- On the leg from $(0,-2)$ to $(0,2)$, $x=0$, so $f(x,y)=y^2$. Since $-2 \le y \le 2$, this has a minimum at $f(0,0)=0$ and a maximum at $f(0,\pm 2)=4$.
- The leg from $(0,-2)$ to $(2,0)$ is part of the line $y=x-2$, so here

  $$f(x,y) = x^2 + (x-2)^2 - 2x = 2x^2 - 6x + 4.$$

  Using single-variable extreme value tools (or the geometry of a parabola), this has a minimum at $x=\frac{3}{2}$, where $y=\frac{3}{2}-2=\frac{-1}{2}$ and $f(\frac{3}{2},\frac{-1}{2})=\frac{-1}{2}$.
- The leg from $(0,2)$ to $(2,0)$ is part of the line $y=2-x$, so here

  $$f(x,y) = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4.$$

  This is the same function of x, so again we have $x=\frac{3}{2}$, which means $y=2-\frac{3}{2}=\frac{1}{2}$ and $f(\frac{3}{2},\frac{1}{2})=\frac{1}{2}$.

The points we have to consider are

$$\begin{aligned} f(1,0) &= -1 \\ f(0,\pm 2) &= 4 \\ f\left(\tfrac{3}{2},\tfrac{-1}{2}\right) &= \tfrac{-1}{2} \\ f\left(\tfrac{3}{2},\tfrac{1}{2}\right) &= \tfrac{1}{2} \end{aligned}$$

The $\boxed{\text{smallest value is } f(1,0)=-1}$ and the $\boxed{\text{largest is } f(0,2)=f(0,-2)=4}$.